FREE FALL OF A MATERIAL POINT IN A SATELLITE CABIN

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1. We will begin the consideration of the problem with the case of free flight of a satellite. The equation of motion of its inertia center C will then be of the form

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r} \tag{1.1}$$

where $\mathbf{r} = \mathbf{OC}$ is the radius vector of point C with the origin at the earth's center; μ is the product of its mass and the gravitational constant.* The forces of nongravitational origin are assumed absent.

The location of the point M falling within the satellite cabin will be defined by the radius-vector $\boldsymbol{\rho} = \mathbf{CM}$; its equation of motion will be

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* We neglect the terms depending on the oblateness of the earth, as well as the terms proportional to the squares of the relations of the principal central radii of inertia to the radius-vector $\mathbf{r}$. In taking into account the latter correction, the following term should be added to the right-hand side of equation (1.1)

$$+\frac{3\mu}{Mr^5}\mathbf{r}\cdot\left(-\frac{1}{2}E\vartheta + \frac{5E}{2r^2}\mathbf{r}\cdot\boldsymbol{\Theta}^c\cdot\mathbf{r} - \boldsymbol{\Theta}^c\right)$$

where $\boldsymbol{\Theta}^c$ is the central tensor of inertia of the satellite, ϑ is the first variant of this tensor (sum of the central moments of inertia), E is the unit tensor and M is the satellite mass. In taking into account this correction, the terms proportional to the attraction of the point by the masses within the satellite should also, apparently, be added to the right-hand side of equation (1.2).

$$\ddot{\mathbf{r}} + \ddot{\boldsymbol{\rho}} = -\frac{\mu}{|\mathbf{r}+\boldsymbol{\rho}|^3}(\mathbf{r}+\boldsymbol{\rho}) \tag{1.2}$$

By eliminating $\ddot{\mathbf{r}}$ with the aid of the equation (1.1) and by retaining only the linear terms in ρ, we obtain the equation

$$\ddot{\boldsymbol{\rho}} = \frac{\mu}{r^3}\left(3\frac{\mathbf{r}\cdot\boldsymbol{\rho}}{r^2}\mathbf{r} - \boldsymbol{\rho}\right) \tag{1.3}$$

whose right-hand side characterizes the nonhomogeneity of the gravitational field within the satellite cabin. This represents a variational equation for the motion of the center of inertia (1.1). Its general integral, containing six constants representing the elliptic elements of the orbit

$$a,\ e,\ t_0,\ \Omega,\ i,\ \omega \tag{1.4}$$

is known. The derivatives of the radius-vector $\mathbf{r}$ with respect to these constants represent, as is known, the solutions of the variational equation (1.3).

Having formed certain linear expressions with respect to these derivatives, we obtain the system of functions

$$\begin{aligned}
\mathbf{q}_1 &= \left[\frac{r}{a} - \frac{3n(t-t_0)}{2\sqrt{1-e^2}}e\sin\varphi\right]\mathbf{e}_r - \frac{3n(t-t_0)}{2\sqrt{1-e^2}}(1+e\cos\varphi)\mathbf{e}_\varphi \\
\mathbf{q}_2 &= -\cos\varphi\,\mathbf{e}_r + \frac{2+e\cos\varphi}{1+e\cos\varphi}\sin\varphi\,\mathbf{e}_\varphi \qquad (\mathbf{e}_\varphi = \mathbf{e}_3\times\mathbf{e}_r) \\
\mathbf{q}_3 &= \sin\varphi\,\mathbf{e}_r + \frac{2+e\cos\varphi}{1+e\cos\varphi}\cos\varphi\,\mathbf{e}_\varphi \qquad (\mathbf{r} = r\mathbf{e}_r) \\
\mathbf{q}_4 &= \frac{r}{a}\mathbf{e}_\varphi, \qquad \mathbf{q}_5 = \frac{r}{a}\cos\varphi\,\mathbf{e}_3, \qquad q_6 = \frac{r}{a}\sin\varphi\,\mathbf{e}_3
\end{aligned} \tag{1.5}$$

Here $\mathbf{e}_r$, $\mathbf{e}_\varphi$, $\mathbf{e}_3$ are unit vectors of the orbital trihedron of the satellite; $\mathbf{e}_3$ is normal to the plane of the orbit.

The expression for the radius-vector r is of the form

$$r = \frac{a(1-e^2)}{1+e\cos\varphi}, \tag{1.6}$$

where φ denotes the true anomaly.

The constants t_0 and n denote the time of passing of the perigee and mean motion (circular orbital frequency of rotation). Taking into account the well known relations of Keplerian motion (1.7)

$$\dot{\varphi} = n\sqrt{1-e^2}\,\frac{a^2}{r^2}, \quad \dot{r} = \frac{nea}{\sqrt{1-e^2}}\sin\varphi, \quad \dot{\mathbf{e}}_r = \dot{\varphi}\mathbf{e}_\varphi, \quad \dot{\mathbf{e}}_\varphi = -\dot{\varphi}\mathbf{e}_r, \quad \dot{\mathbf{e}}_3 = 0$$

we construct now the expressions for time derivatives of the vectors $\mathbf{q}_s$

$$\begin{gathered}
\dot{\mathbf{q}}_1 = -n\left\{\left[\frac{e\sin\varphi}{2\sqrt{1-e^2}} - \frac{3}{2}n(t-t_0)\frac{a^2}{r^2}\right]\mathbf{e}_r + \frac{1+e\cos\varphi}{2\sqrt{1-e^2}}\mathbf{e}_\varphi\right\} \\
\dot{\mathbf{q}}_2 = \frac{n}{\sqrt{1-e^2}}\frac{a}{r}\left(-\sin\varphi\mathbf{e}_r + \frac{e+\cos\varphi}{1+e\cos\varphi}\mathbf{e}_\varphi\right) \\
\dot{\mathbf{q}}_3 = -\frac{n}{\sqrt{1-e^2}}\frac{a}{r}\left(\cos\varphi\mathbf{e}_r + \frac{\sin\varphi}{1+e\cos\varphi}\mathbf{e}_\varphi\right) \\
\dot{\mathbf{q}}_4 = \frac{n}{\sqrt{1-e^2}}[(1+e\cos\varphi)\mathbf{e}_r + e\sin\varphi\mathbf{e}_\varphi] \\
\dot{\mathbf{q}}_5 = \frac{-n}{\sqrt{1-e^2}}\sin\varphi\mathbf{e}_3, \quad \dot{\mathbf{q}}_6 = \frac{n}{\sqrt{1-e^2}}(\cos\varphi + e)\mathbf{e}_3
\end{gathered} \tag{1.8}$$

The vectors $\mathbf{q}_s$ and $\dot{\mathbf{q}}_s$ form a system of integrals for the equation of motion (1.3) of the point M. The square 6 x 6 matrix for the projection of these vectors on the axes of the orbital trihedron can in the following analysis be replaced by two matrices of 4 x 4 and 2 x 2

$$\left\|\begin{matrix}
\mathbf{q}_1\cdot\mathbf{e}_r & \mathbf{q}_2\cdot\mathbf{e}_r & \mathbf{q}_3\cdot\mathbf{e}_r & \mathbf{q}_4\cdot\mathbf{e}_r \\
\mathbf{q}_1\cdot\mathbf{e}_\varphi & \mathbf{q}_2\cdot\mathbf{e}_\varphi & \mathbf{q}_3\cdot\mathbf{e}_\varphi & \mathbf{q}_4\cdot\mathbf{e}_\varphi \\
\dot{\mathbf{q}}_1\cdot\mathbf{e}_r & \dot{\mathbf{q}}_2\cdot\mathbf{e}_r & \dot{\mathbf{q}}_3\cdot\mathbf{e}_r & \dot{\mathbf{q}}_4\cdot\mathbf{e}_r \\
\dot{\mathbf{q}}_1\cdot\mathbf{e}_\varphi & \dot{\mathbf{q}}_2\cdot\mathbf{e}_\varphi & \dot{\mathbf{q}}_3\cdot\mathbf{e}_\varphi & \dot{\mathbf{q}}_4\cdot\mathbf{e}_\varphi
\end{matrix}\right\| = \alpha, \qquad \left\|\begin{matrix}
\mathbf{q}_5\cdot\mathbf{e}_3 & \mathbf{q}_6\cdot\mathbf{e}_3 \\
\dot{\mathbf{q}}_5\cdot\mathbf{e}_3 & \dot{\mathbf{q}}_6\cdot\mathbf{e}_3
\end{matrix}\right\| = \beta \tag{1.9}$$

The determinants $|\alpha|$ and $|\beta|$ of these matrices are the Wronskians of two systems of linear equations of fourth and second orders, obtained by projecting the vector equation (1.3) on the axes of the orbital trihedron; the diagonal elements in the matrices of the coefficients of the right-hand sides of these equations are absent and therefore the Wronskians are constant; they can be easily computed by letting $t = t_0$ and $\varphi = 0$. We find

$$|\alpha| = -\tfrac{1}{2}n^2, \qquad |\beta| = n\sqrt{1-e^2} \tag{1.10}$$

Thus, the solutions (1.5) and (1.8) are linearly independent; they remain such also in the case of a circular orbit (for $e = 0$). This was obtained by proper choice of the linear combinations of the derivatives of the radius-vector $\mathbf{r}$ with respect to the constants (1.4).

In the following the inverse matrices will be required

$$\gamma = \alpha^{-1}, \qquad \delta = \beta^{-1} \tag{1.11}$$

The computation of γ is somewhat unwieldy but it can be partly avoided by considering the existence of the coupling of the elements in the third and fourth rows of this matrix with the derivatives of elliptic elements which are known in the theory of perturbation of elliptic elements. The expressions for the matrices γ and δ are given in the Appendix.

The vectors ρ and $\dot{\rho}$ defining the position and velocity of the falling point M in the cabin of the satellite fixed in a moving coordinate system with the origin at the center of inertia C are now represented in the form

$$\rho = \sum_{s=1}^{6} C_s \mathbf{q}_s, \qquad \dot{\rho} = \sum_{s=1}^{6} C_s \dot{\mathbf{q}}_s \tag{1.12}$$

We will assume that the angular velocity vector ω of the $Cxyz$ system attached to the satellite is known. Then, denoting by ρ^* the velocity vector of the point M relative to these axes (relative to the cabin), we have

$$\dot{\rho} = \overset{*}{\rho} + \omega \times \rho \tag{1.13}$$

Therefore, assuming here and in the following, that the separation of the point M from the cabin occurs with a zero relative velocity, we obtain the initial conditions

$$t = t_*, \qquad \rho = \rho^\circ, \qquad \dot{\rho} = \omega^\circ \times \rho^\circ \tag{1.14}$$

We will note that during such shockless separation of the point M the angular velocity vector ω remains continuous. The expressions for the constants C_s in the solution (1.12) are now written in the form

$$C_s = \gamma_{s1}{}^\circ \rho^\circ \cdot \mathbf{e}_r{}^\circ + \gamma_{s2}{}^\circ \rho^\circ \cdot \mathbf{e}_\varphi{}^\circ + \gamma_{s3}{}^\circ (\omega^\circ \times \rho^\circ) \cdot \mathbf{e}_r{}^\circ + \gamma_{s4}{}^\circ (\omega^\circ \times \rho^\circ) \cdot \mathbf{e}_\varphi{}^\circ$$
$$(s = 1, 2, 3, 4) \tag{1.15}$$
$$C_{4+k} = \delta_{k1}{}^\circ \rho^\circ \cdot \mathbf{e}_3 + \delta_{k2}{}^\circ (\omega^\circ \times \rho^\circ) \cdot \mathbf{e}_3 \qquad (k = 1, 2)$$

A relatively simple expression for the vector ρ is obtained in the particular case when the separation of the point M from the cabin occurs at the instant of passing through the perigee ($t_* = t_0$, $\varphi = 0$) and the satellite is at that instant stabilized in the orbital axes ($\omega^\circ = \mathbf{e}_3 \dot{\varphi}^\circ$). Then

$$\rho = \left[\frac{2(2+e)}{1-e^2}\mathbf{q}_1 + \frac{3(1+e)}{1-e}\mathbf{q}_2\right]\rho^\circ \cdot \mathbf{e}_r{}^\circ + \frac{1}{1-e}\mathbf{q}_4 \rho^\circ \cdot \mathbf{e}_\varphi{}^\circ + \frac{(1+e)\cos\varphi}{1+e\cos\varphi}\rho^\circ \cdot \mathbf{e}_3\mathbf{e}_3 \tag{1.16}$$

while in the case of a circular orbit ($e = 0$, $\varphi = nt$)

$$\begin{aligned}\rho - \rho_0 = \mathbf{e}_3 \times \rho^\circ \sin\varphi - \rho^\circ (1 - \cos\varphi) + \\ + 3\rho^\circ \cdot \mathbf{e}_r{}^\circ \times [(1 - \cos\varphi)\mathbf{e}_r - 2(\varphi - \sin\varphi)\mathbf{e}_\varphi]\end{aligned} \tag{1.17}$$

For the circular orbit, in the general case (i.e. for $\omega^\circ \neq \mathbf{e}_3 n$) we get (it may be assumed $t_* = 0$)

$$\begin{aligned}\rho - \rho^\circ = \frac{1}{n}\omega^\circ \times \rho^\circ \sin\varphi + \\ + (1-\cos\varphi)\Big\{-\rho^\circ + \Big[\rho^\circ \cdot \mathbf{e}_r{}^\circ + 2\Big(\frac{\omega^\circ}{n} \times \rho^\circ\Big)\cdot \mathbf{e}_\varphi{}^\circ\Big]\mathbf{e}_r - \\ - 2\Big[\rho^\circ \cdot \mathbf{e}_\varphi{}^\circ + \frac{1}{n}(\omega^\circ \times \rho^\circ)\cdot \mathbf{e}_r{}^\circ\Big]\mathbf{e}_\varphi\Big\} - \\ - 3(\varphi - \sin\varphi)\Big[\rho^\circ \cdot \mathbf{e}_r{}^\circ + \frac{1}{n}(\omega^\circ \times \rho^\circ)\cdot \mathbf{e}_\varphi{}^\circ\Big]\mathbf{e}_\varphi + \\ + \sin\varphi\Big\{\Big[\rho^\circ \cdot \mathbf{e}_\varphi{}^\circ + \frac{1}{n}(\omega^\circ \times \rho^\circ)\cdot \mathbf{e}_r{}^\circ\Big](\mathbf{e}_r{}^\circ - \mathbf{e}_r) - \\ - \Big[\rho^\circ \cdot \mathbf{e}_r{}^\circ - \frac{1}{n}(\omega^\circ \times \rho^\circ)\cdot \mathbf{e}_\varphi{}^\circ\Big](\mathbf{e}_\varphi - \mathbf{e}_\varphi{}^\circ)\Big\}\end{aligned} \tag{1.18}$$

For $\omega^\circ = \mathbf{e}_3 n$ we return to the expression (1.17); in the inertially stabilized satellite (for $t = 0$) $\omega^\circ = 0$ and the first order terms relative to $\varphi = nt$ are omitted.

Neglecting the nonhomogeneity of the gravitational field within the satellite, and for the same initial conditions (1.14) we would obtain

$$\ddot{\rho} = 0, \qquad \rho - \rho^\circ = \frac{1}{n}\omega^\circ \times \rho^\circ nt \tag{1.19}$$

A comparison with (1.18) shows that the influence of the gravitational nonhomogeneity is reflected in the terms of order $(nt)^2$.

Returning to (1.18), let the inertial axes coincide with the orbital axes at time $t = 0$ (i.e. in the directions $\mathbf{e}_r{}^\circ$, $\mathbf{e}_\varphi{}^\circ$, $\mathbf{e}_3$) and denote by ψ, ϑ, χ the Euler angles defining the directions of the system $Cxyz$ relative to the inertial system. Here ψ (pitch) is the angle of rotation about the normal to the orbit plane $\mathbf{e}_3$, ϑ (yaw) is the angle of rotation about the displaced axis $\mathbf{e}_r{}^\circ$ (local vertical in the satellite before rotation), χ (roll) is the angle of rotation about the twice displaced axis $\mathbf{e}_\varphi{}^\circ$: the latter direction defines the axis Cx, while the axes Cy and Cz will be along the directions of $\mathbf{e}_3$ and $\mathbf{e}_r$. Assuming the Euler angles small, we obtain the following table of direction cosines:

(1.20)

	x	y	z
$\mathbf{e}_\varphi{}^\circ$	1	$-\vartheta$	ψ
$\mathbf{e}_3{}^\circ$	ϑ	1	$-\chi$
$\mathbf{e}_r{}^\circ$	$-\psi$	χ	1

	x	y	z
$\mathbf{e}_\varphi$	$\cos\varphi + \psi\sin\varphi$	$-\vartheta\cos\varphi - \chi\sin\varphi$	$\psi\cos\varphi - \sin\varphi$
$\mathbf{e}_3$	ϑ	1	$-\chi$
$\mathbf{e}_r$	$\sin\varphi - \psi\cos\varphi$	$-\vartheta\sin\varphi + \chi\cos\varphi$	$\cos\varphi + \psi\sin\varphi$

and the expression for the angular velocity vector

$$\frac{1}{n}\omega = \left(\frac{\dot{\chi}}{n} + \vartheta\right)\mathbf{i}_x + \left(\frac{\dot{\psi}}{n} + 1\right)\mathbf{i}_y + \left(\frac{\dot{\vartheta}}{n} - \chi\right)\mathbf{i}_z \qquad (1.21)$$

Now, it is easy to express the running coordinates x, y, z (projections of ρ) of the falling point in terms of its initial coordinates x_0, y_0, z_0. These formulas will contain the angles ψ, ϑ, χ, their initial values (at the time of separation from the cabin), and the initial values of their derivatives.

2. A more difficult problem of the falling point in a satellite results when nongravitational forces also act on the satellite. The acceleration caused by these forces will be denoted by **f**. Then the equation of motion of the satellite center of inertia will be

$$\ddot{\mathbf{r}}^* = -\frac{\mu}{r^{*3}}\mathbf{r}^* + \mathbf{f} \qquad (2.1)$$

The equation of motion of the point M will, of course, remain of the form (1.2) while upon eliminating $\ddot{\mathbf{r}}^*$, in place of (1.3) we will obtain the equation

$$\ddot{\rho} = \frac{\mu}{r^{*}_{3}}\left(3\,\frac{\mathbf{r}^* \cdot \rho}{r^2}\mathbf{r}^* - \rho\right) - \mathbf{f} \qquad (2.2)$$

Retaining the terms in the first group in (2.2), the order of magnitude of which is $g|\rho|/r \approx 10^{-6} - 10^{-7}$ g, is meaningful only for sufficiently small values of $|\mathbf{f}|$, i.e. for the forces in the order of, say, 1-10 g per ton. But for such forces as well as for forces exceeding them by several orders of magnitude (say 1 kg per ton), the **f**-term on the right-hand side of (2.1) will be quite small compared to the force of gravity and, consequently, for small intervals of time (one or two revolutions) it is permissible to consider f as a term perturbing the basic Keplerian motion of the center of inertia of the satellite. Therefore, letting

$$\mathbf{r}^* = \mathbf{r} + \delta\mathbf{r} \qquad (2.3)$$

where **r** defines the Keplerian motion and satisfies the differential equation (1.1), we obtain the nonhomogeneous linear differential equation

$$(\delta\mathbf{r})^{\cdot\cdot} = \frac{\mu}{r^3}\left(3\,\frac{\mathbf{r} \cdot \delta\mathbf{r}}{r^2}\mathbf{r} - \delta\mathbf{r}\right) + \mathbf{f} \qquad (2.4)$$

in which **f** is given for the Keplerian (and not actual) orbit of the satellite inertia center.

The general solution of the corresponding homogeneous equation is known. Utilizing the method of the variation of parameters, we let

$$\delta\mathbf{r} = \sum_{s=1}^{6} D_s(t)\,\mathbf{q}_s, \qquad (\delta\mathbf{r})^{\cdot} = \sum_{s=1}^{6} D_s(t)\,\dot{\mathbf{q}}_s \tag{2.5}$$

and in order to determine $D_s(t)$ we obtain the system of linear equations

$$\sum_{s=1}^{6} \dot{D}_s(t)\,\mathbf{q}_s = 0. \qquad \sum_{s=1}^{6} \dot{D}_s(t)\,\dot{\mathbf{q}}_s = \mathbf{f} \tag{2.6}$$

With the aid of the matrices γ and δ we obtain (2.7)

$$D_s(t) = \int_{t_*}^{t} (\gamma_{s3}\mathbf{e}_r + \gamma_{s4}\mathbf{e}_\varphi)\cdot \mathbf{f}\,dt \quad (s=1,2,3,4), \quad D_{4+k} = \int_{t_*}^{t} \delta_{k2}\mathbf{f}\cdot \mathbf{e}_3\,dt \quad (k=1,2)$$

and therefore

$$\delta\mathbf{r} = 0, \qquad (\delta\mathbf{r})^{\cdot} = 0 \qquad \text{for } \;= t_* \tag{2.8}$$

Let us now consider the geometric sum $\boldsymbol{\rho} + \delta\mathbf{r}$; then, neglecting the terms of the order of the product $|\boldsymbol{\rho}|\,|\delta\mathbf{r}|$, we obtain from the equations (2.2) and (2.4) the homogeneous linear equation

$$\ddot{\boldsymbol{\rho}} + (\delta\mathbf{r})^{\cdot\cdot} = \frac{\mu}{r^3}\left[3\,\frac{\mathbf{r}}{r^2}\,\mathbf{r}\cdot(\boldsymbol{\rho}+\delta\mathbf{r}) - (\boldsymbol{\rho}+\delta\mathbf{r})\right] \tag{2.9}$$

the solution of which in the form of (1.12) is known. Thus

$$\boldsymbol{\rho} = \sum_{s=1}^{6} \mathbf{q}_s\, C_s - \delta\mathbf{r} \tag{2.10}$$

and in view of (2.8) the constants C_s are determined by the same formulas (1.15) as before.

The most interesting case is that of aerodynamic resistance. The acceleration created by this force can be of the same order of magnitude as the acceleration resulting from the nonhomogeneity of the gravitational field so that the application of the derived relations is fully justified.

The expression for $\mathbf{f}$ is here given in the form

$$\mathbf{f} = -\lambda(r,v)\,\mathbf{v} = -\frac{na}{\sqrt{1-e^2}}\,\lambda(r,v)\,[e\sin\varphi\,\mathbf{e}_r + (1+e\cos\varphi)\,\mathbf{e}_\varphi] \tag{2.11}$$

In the case of a circular orbit a very simple calculation with the above derived formulas yields

$$\rho = \rho^* + \frac{2\lambda\,(a,\,an)}{n}\,a\left[\mathbf{e}_r(\varphi - \sin\varphi) + 2\mathbf{e}_\varphi\left(1 - \cos\varphi - \frac{3}{8}\varphi^2\right)\right] \qquad (2.12)$$

where ρ^* is determined from the formula (1.18). Neglecting in (2.2) the terms due to the nonhomogeneity of the gravitational field, we have

$$\ddot{\rho} = -\mathbf{f}, \qquad \rho = \rho^* - \int_{t_*}^{t} (t - t')\,\mathbf{f}(t')\,dt' \qquad (2.13)$$

where ρ^* is given by (1.19). Replacing the vector $\mathbf{f}(t')$ by its projections on the orbital trihedron of the satellite center of inertia and noting that the axes of the point t are obtained by rotating the axes at the point t' by an angle $\varphi - \varphi'$ about $\mathbf{e}_3$, we express (2.13) in the form

$$\rho = \rho^* - \mathbf{e}_r \int_{t_*}^{t} (t - t')\,[\mathbf{f}(t')\cdot\mathbf{e}_r'\cos(\varphi - \varphi') + \mathbf{f}(t')\cdot\mathbf{e}_\varphi'\sin(\varphi - \varphi')]\,dt' - {}$$
$$- \mathbf{e}_\varphi \int_{t_*}^{t} (t - t')\,[-\mathbf{f}(t')\cdot\mathbf{e}_r'\sin(\varphi - \varphi') + \mathbf{f}(t')\cdot\mathbf{e}_\varphi'\cos(\varphi - \varphi')]\,dt' \qquad (2.14)$$

For example, in the case of the aerodynamic force (2.11) and circular orbital motion

$$\rho = \rho^* + \frac{a\lambda\,(a,\,an)}{n}\,[\mathbf{e}_r(\sin\varphi - \varphi\cos\varphi) + \mathbf{e}_\varphi(\varphi\sin\varphi + \cos\varphi - 1)] \qquad (2.15)$$

The lowest terms of the series expansion in the powers of φ, resulting from the presence of the force $\mathbf{f}$, coincide with those obtained from the formula (2.12). The nonhomogeneity of the gravitational field within the space of the cabin is in evidence during prolonged falls.

Appendix. **The elements of the matrices γ and δ are**

$$\gamma_{11} = 2\frac{a^2}{r^2}, \qquad \gamma_{12} = 0, \qquad \gamma_{13} = \frac{2e\sin\varphi}{n\sqrt{1 - e^2}}, \qquad \gamma_{14} = \frac{2(1 + e\cos\varphi)}{n\sqrt{1 - e^2}}$$

$$\gamma_{21} = \frac{(1 + e\cos\varphi)(e + \cos\varphi)}{1 - e^2}, \qquad \gamma_{22} = \sin\varphi, \qquad \gamma_{23} = \frac{\sqrt{1 - e^2}}{n}\sin\varphi,$$

$$\gamma_{24} = \frac{\sqrt{1 - e^2}}{n}\,\frac{e + 2\cos\varphi + e\cos^2\varphi}{1 + e\cos\varphi}$$

$$\gamma_{31} = \frac{3n(t - t_0)}{\sqrt{1 - e^2}}\,\frac{a^2}{r^2}\,e - \frac{1 + e\cos\varphi + e^2}{1 - e^2}\sin\varphi, \qquad \gamma_{32} = \cos\varphi,$$

$$\gamma_{33} = \frac{1}{n}\left[3\,\frac{n(t-t_0)}{1-e^2}\,e^2\sin\varphi - \frac{\sqrt{1-e^2}}{1+e\cos\varphi}\,(2e-\cos\varphi-e\cos\varphi)\right]$$

$$\gamma_{34} = \frac{1}{n}\left[-\frac{\sqrt{1-e^2}}{1+e\cos\varphi}\,(2+e\cos\varphi)\sin\varphi + \frac{3n(t-t_0)}{1-e^2}\,(1+e\cos\varphi)\,e\right]$$

$$\gamma_{41} = \frac{3n(t-t_0)}{(1-e^2)^{3/2}}\,\frac{a^2}{r^2} - e\sin\varphi\,\frac{2+e\cos\varphi}{(1-e^2)^2}\,,\qquad \gamma_{42} = \frac{-1}{1-e^2}$$

$$\gamma_{43} = \frac{1}{n}\left[3\,\frac{n(t-t_0)}{(1-e^2)^2}\,e\sin\varphi + \frac{e\cos\varphi+e^2\cos^2\varphi-2}{\sqrt{1-e^2}\,(1+e\cos\varphi)}\right],$$

$$\gamma_{44} = \frac{1}{n}\left[-e\sin\varphi\,\frac{2+e\cos\varphi}{\sqrt{1-e^2}\,(1+e\cos\varphi)} + 3\,\frac{n(t-t_0)}{(1-e^2)^2}\,(1+e\cos\varphi)\right]$$

$$\delta_{11} = \frac{\cos\varphi+e}{1-e^2}\,,\qquad \delta_{12} = -\frac{\sqrt{1-e^2}\sin\varphi}{n(1+e\cos\varphi)}\,,\qquad \delta_{21} = \frac{\sin\varphi}{1-e^2}\,,\qquad \delta_{22} = \frac{\sqrt{1-e^2}\cos\varphi}{n(1+e\cos\varphi)}$$

Denoting

$$x_1 = \mathbf{q}_s\cdot\mathbf{e}_r,\quad x_2 = \mathbf{q}_s\cdot\mathbf{e}_\varphi,\quad x_3 = \dot{\mathbf{q}}_s\cdot\mathbf{e}_r,\quad x_4 = \dot{\mathbf{q}}_s\cdot\mathbf{e}_\varphi,\quad x_5 = \mathbf{q}_s\cdot\mathbf{e}_3,\quad x_6 = \dot{\mathbf{q}}_s\cdot\mathbf{e}_3$$

the vector equation (1.3) will be represented by the system of equations

$$\dot{x}_1 = \dot{\varphi}\,x_2 + x_3,\qquad \dot{x}_3 = \dot{\varphi}x_4 + \frac{2\mu}{r^3}\,x_1,\qquad \dot{x}_5 = x_6$$

$$\dot{x}_2 = -\dot{\varphi}x_1 + x_4,\qquad \dot{x}_4 = -\dot{\varphi}x_3 - \frac{\mu}{r^3}\,x_2,\qquad \dot{x}_6 = -\frac{\mu}{r_3}\,x_5$$

The conjugate system is of the form

$$\dot{y}_1 = \dot{\varphi}y_2 - \frac{2\mu}{r^3}\,y_3,\qquad \dot{y}_3 = -y_1 + \dot{\varphi}y_4,\qquad \dot{y}_5 = \frac{\mu}{r^3}\,y_6$$

$$\dot{y}_2 = -\dot{\varphi}y_1 + \frac{\mu}{r^3}\,y_4,\qquad \dot{y}_4 = -y_2 - \dot{\varphi}y_3,\qquad \dot{y}_6 = -y_5$$

and $\gamma_{sr} = y_r$ $(s = 1, 2, 3, 4)$ will be its solutions. This can be used to verify the calculations.

Translated by V.C.